

# THE WONG-ROSAY TYPE THEOREM FOR KÄHLER MANIFOLDS

BINGYUAN LIU

**ABSTRACT.** The Wong-Rosay theorem characterizes the strongly pseudoconvex domains of  $\mathbb{C}^n$  by their automorphism groups. It has a lot of generalizations to other kinds of domains (for example, the weakly pseudoconvex domains). However, most of them are for domains of  $\mathbb{C}^n$ . In this note, we generalize the Wong-Rosay theorem to the simply-connected complete Kähler manifold with a negative sectional curvature. One aim of this note is to exhibit a Wong-Rosay type theorem of manifolds with holomorphic non-invariant metrics.

## 0. INTRODUCTION

Let  $X$  be a Hadamard manifold, i.e. a simply-connected complete Riemannian manifold  $(M, g)$  with non-positive sectional curvature everywhere. It is well-known that one can define a boundary  $\partial X$  artificially to make  $X \cup \partial X$  compact, which has been done by [6]. We will try to explain the procedure with one of the equivalent methods (curious readers are referred to [1]).

Two unit speed geodesics  $\gamma_i : \mathbb{R} \rightarrow X$  are called asymptotic, if there is a constant  $c \in \mathbb{R}$ , such that  $d(\gamma_1(t), \gamma_2(t)) \leq c$  for all  $t \geq 0$ . It is not hard to see ‘asymptotic’ determines an equivalence relation. We define the asymptotic boundary  $\partial X$  to be the set of equivalence classes of this relation. The elements of  $\partial X$  are also called points at infinity.

Let  $\overline{X} := X \cup \partial X$  which makes  $X$  a compact topological manifold-with-boundary, in which the topology is defined as follows. Let  $z$  be a point at infinity and  $C_x(z, \epsilon) := \{y \in \overline{X} \mid y \neq x, \angle_x(z, y) < \epsilon\}$ . Here,  $\angle_x(z, y)$  is defined to be the angle formed by  $\gamma'_1(0)$  and  $\gamma'_2(0)$ , where  $\gamma_1$  and  $\gamma_2$  are geodesics (with unit speed) joining  $x$  with  $z$  and  $y$  respectively. Then  $\{C_x(z, \epsilon)\}_{x, \epsilon}$  forms a base of neighborhoods around  $z$  in  $\overline{X}$ . Also,  $\{C_x(z, \epsilon) \cap \partial X\}_{x, \epsilon}$  forms an open neighborhood of  $z$  in  $\partial X$ . One should note that, in Hadamard manifolds, the two geodesics tend to diverge faster or at least at the same speed as in the Euclidean situation by comparison theorem. On the other hand, it is also clear that for any point  $z \in \partial X$  at infinity and  $x \in X$ , there is a unique unit speed geodesic  $\gamma : \mathbb{R} \rightarrow X$  with  $\gamma(0) = x$  and  $\lim_{t \rightarrow \infty} \gamma(t)$  falls in the class  $[z]$ . We refer the reader to [1] and [4].

Let  $M$  be a  $m$ -dimensional simply-connected complete Kähler manifold with a negative sectional curvature. Then  $M$  has to be a Hadamard manifold. Thus,  $M$  has an asymptotic boundary and we denote it  $\partial M$ . Furthermore, we assume  $M$  has a constant negative holomorphic sectional curvature in a neighborhood  $U$  of a point  $p \in \partial M$  at infinity in  $\overline{M}$ .

We also denote  $\mathbb{B}^m$  the  $m$ -dimensional unit ball in  $\mathbb{C}^m$ .

---

*Date:* July 17, 2014.

**Theorem 0.1.** *Let  $M$  be a simply-connected complete Kähler manifold with a negative sectional curvature. Suppose, also,  $M$  has a constant negative holomorphic sectional curvature in a neighborhood  $U$  of a point  $p \in \partial M$  at infinity in  $\overline{M}$ . If we assume there is a family of biholomorphic maps  $\phi_j$  so that for any  $q \in M$ ,  $\phi_j(q) \rightarrow p$ , as  $j \rightarrow \infty$ , then  $M$  admits a biholomorphic imbedding into the unit ball  $\mathbb{B}^m$ .*

*Remark 1.* Without loss of the generality, we can always assume  $M$  has a constant holomorphic sectional curvature  $-1$  in a neighborhood  $U$  of a point  $p \in \partial M$  at infinity.

Theorem 0.1 proves a generalization of the Wong-Rosay theorem in several complex variables (see Corollary 2.1). The Wong-Rosay theorem states, in  $\mathbb{C}^n$ , if  $\Omega \subset \mathbb{C}^n$  admits a boundary point  $p \in \partial\Omega$  as a strongly pseudoconvex point and an interior point  $q$  such that  $\phi_j(q) \rightarrow p$  as  $j \rightarrow \infty$  where  $\phi_j \in \text{Aut}(\Omega)$  are a sequence of holomorphic automorphisms, then  $\Omega$  is biholomorphic to unit ball  $\mathbb{B}^n$ .

The Wong-Rosay theorem has a lot of generalizations in  $\mathbb{C}^n$  by many mathematicians to a weakly pseudoconvex point  $p \in \partial\Omega$ . The limit point  $p$  of  $\{\phi_j(q)\}$  is called an orbit accumulation point. One should notice that, the generalizations of the Wong-Rosay theorem usually are very difficult compared with the original one. Until now, only several partial results are known. This is because the rigidity of biholomorphic maps. Moreover, the reader should be reminded that the biholomorphic maps are quite different from the diffeomorphisms. For example, it is well-known that any non-empty open star domain (or specifically a non-empty open convex subset) of  $\mathbb{R}^n$  is diffeomorphic to a unit ball. However, it is clear that there exist a lot of star domains in  $\mathbb{C}^n$  which are not biholomorphic to a ball. The following well-known example explains that star-domains  $\Omega$  do not need to be biholomorphic to a ball.

**Example 0.1.** Let  $\Omega = \{(z, w) : |z|^4 + |w|^4 < 1\} \subset \mathbb{C}^2$ , and then there does not exist an interior point  $q \in \Omega$ , a family of biholomorphism  $\phi_j$  and a boundary point  $p \in \partial\Omega$  such that  $\phi_j(q) \rightarrow p$  as  $j \rightarrow \infty$ . That is  $\text{Aut}(\Omega)$  is compact. Since  $\text{Aut}(\mathbb{B}^m)$  is noncompact where  $\mathbb{B}^m$  denotes the unit ball in  $\mathbb{C}^m$ , we can see  $\Omega$  is not biholomorphic to  $\mathbb{B}^m$ .

For the proof of compactness of  $\text{Aut}(\Omega)$ , please see [8]. Many mathematicians also studied the automorphism group  $\text{Aut}(\Omega)$  to approach the classification from the point of view of Lie groups based on the famous theorem of Cartan:  $\text{Aut}(\Omega)$  is finite dimensional (real) Lie group if  $\Omega$  is bounded (The curious reader is referred to [13]).

The Wong-Rosay theorem and its generalizations have been studied by many experts including, but not limited to: Bedford, Berteloot, Cheung, Fu, Gaussier, Greene, Isaev, Kaup, Kim, Klembeck, Kodama, Krantz, Landucci, Pinchuk, Seshadri, Verma and Wong. The study of the Wong-Rosay theorem for Kähler manifolds dates back to the 1970s (see, e.g., [15] and [14] for Bergman metric). However, most of the Kähler metrics they used are of invariance under biholomorphic maps. That means, the Kähler metric that the manifold is endowed with is not arbitrary. Recently, Seshadri and Verma studied the property of a Kähler manifold equipped with an arbitrary (including non-invariant) metric in [18]. The reader is also referred, for general discussion, to [21], [10], [11], [12], [9], [7], [5], [3], [16], [20].

The current short note provides more evidence that the automorphism groups can be extended to an arbitrary Kähler metric. Understanding it is also good for understanding the difference between the general Kähler metric and the Bergman metric.

### 1. THE PROOF OF THEOREM 0.1

The proof can be divided into several steps. We write each step as one lemma as follows.

**Lemma 1.1.** *Let  $M$  be a simply-connected complete Kähler manifold with a negative sectional curvature. Suppose, also,  $M$  has a constant negative holomorphic sectional curvature in a neighborhood  $U$  of a point  $p \in \partial M$  at infinity in  $\overline{M}$ . Then there exists  $C_x(p, \epsilon) \subset U$  for some  $x \in U$  and  $\epsilon > 0$  such that  $C_x(p, \epsilon)$  is bi-holomorphic to a subset  $V$  of  $\mathbb{B}^m$  via  $f$ . Moreover,  $f$  extends to the neighborhood  $C_x(p, \epsilon) \cap \partial M$  of  $p$  in  $\partial M$  which gives a local diffeomorphism from  $C_x(p, \epsilon) \cap \partial M$  into the unit sphere  $\mathbb{S}^m$ .*

*Proof.* This proof is very similar to the one in [18]. For completeness, we include the proof here. We define a smooth map  $\psi$  in order to extend the holomorphic map to the boundary. Let  $\psi$  be a diffeomorphism from  $\mathbb{C}^m$  to  $\mathbb{B}^m$  defined as  $\frac{\pi z}{2} \arctan \|z\|$ , where  $z \in \mathbb{C}^m$ . We also define the modified exponential map  $\widetilde{\exp}_p : \mathbb{B}^m \rightarrow M$  as  $\exp_p \circ \psi^{-1}$ . One can easily see this modified exponential map actually extends homeomorphically to a map from  $\mathbb{B}^m$  onto  $\overline{M}$ . We will still use  $\widetilde{\exp}_p$  to denote this extension.

We now take a neighborhood  $C_x(p, \epsilon) \subset U$  of  $p$  for some  $x \in U$  and  $\epsilon > 0$ . Let  $L$  be a complex linear isometry  $L : T_x M \rightarrow T_0 \mathbb{B}^m$  (see [18]). Hence, we have  $f = \widetilde{\exp}_0 \circ L \circ \widetilde{\exp}_x^{-1} : C_x(p, \epsilon) \rightarrow \overline{\mathbb{B}^m}$  (Here, we consider  $\mathbb{B}^m$  as a Kähler manifold with a negative sectional curvature.). It is clear that  $f = \widetilde{\exp}_0 \circ L \circ \widetilde{\exp}_x^{-1} = \exp_0 \circ L \circ \exp_x$ . It means even if  $\widetilde{\exp}_x$  is not holomorphic due to the term brought by  $\psi$ , the composition  $f$  is holomorphic. Since  $C_x(p, \epsilon)$  is the subset of a Kähler manifold with a constant holomorphic sectional curvature, we obtain  $f$  is a holomorphic (local) isometry.

For the following, we need to check in the neighborhood of  $p$  of  $\partial M$ , its  $CR$  structure is spherical.

On the boundary, we defined the charts as  $f(C_x(p, \epsilon) \cap \partial M)$ . We first check if the transition functions are compatible each other. Suppose there is another  $C_y(p, \xi)$  and let  $g = \widetilde{\exp}_0 \circ L \circ \widetilde{\exp}_y^{-1}$ . It is clear that  $f \circ g^{-1}$  is holomorphic. Since  $f(\partial C_x(p, \epsilon))$  and  $g(\partial C_y(p, \xi))$  are parts of a sphere which is strongly pseudoconvex. Now by the theorem of [2],  $f \circ g^{-1}$  extends to the boundary smoothly. Also, it is clear that the coordinate chart  $(C_x(p, \epsilon) \cap \partial M, f)$  defines a  $CR$ -structure on  $\partial M$  locally around  $p$ . Then  $C_x(p, \epsilon) \cap \partial M$  is locally spherical and this finishes the proof.  $\square$

**Lemma 1.2.** *Let  $M$  be a simply-connected complete Kähler manifold with a negative sectional curvature. Suppose, also,  $M$  has a constant negative holomorphic sectional curvature in a neighborhood  $U$  of a point  $p \in \partial M$  at infinity in  $\overline{M}$ . Assume there is a family of biholomorphic maps  $\phi_j$  and  $q \in M$  such that  $\phi_j(q) \rightarrow p$  as  $j \rightarrow \infty$ . Then  $\{\phi_j\}_{j=1}^\infty$  converges uniformly on arbitrary compact subsets.*

To prove Lemma 1.2, we need to define a new topology around  $p$  instead of the usual topology  $\|z - p\|$ , which is not well-defined. One observes that  $S_1(z) = \log \|z - p\|_{U_0}$  is a local plurisubharmonic function defined in  $U_0 := C_x(p, \epsilon) \setminus \partial M$  as in Lemma 1.1 with value  $-\infty$  at  $p$ , where  $\|\cdot\|_{U_0}$  is the metric induced by the Euclidean metric of  $\overline{\mathbb{B}^m}$  (consider  $\mathbb{B}^m$  as a Euclidean ball now). We now construct a global function defined on  $M$ . Choose two neighborhoods  $\mathfrak{V} \subset \mathfrak{U}$  of  $p$  inside  $U_0$  and a constant  $d > 0$  such that  $\sup\{S_1(z) : z \in U_0 \cap \mathfrak{U}\} = -d$  and  $\mathfrak{V} = \{z \in U_0 \cap \mathfrak{U} : S_1(z) < -2d\}$ .

$$S(z) = \begin{cases} S_1(z) & \text{if } z \in M \cap \overline{\mathfrak{V}} \\ \min(S_1(z), -\frac{d}{2}) & \text{if } z \in M \cap (\mathfrak{U} \setminus \mathfrak{V}) \\ -\frac{d}{2} & \text{otherwise.} \end{cases}$$

We define the base of open subsets around  $p$  as the  $\{z \in \Omega : S(z) < -\alpha\}$  for arbitrary  $\alpha > 0$ . Moreover, we need another topology induced by a local peak function in  $U_0$ .

We construct a global peak function as follows. Since  $U_0$  is a ball and  $\partial U_0$  is  $CR$  diffeomorphism with a sphere, around  $p$  there is a local peak function. Namely, a plurisubharmonic function  $H_1$  defined in  $U_0$  such that  $|H_1(p)| = 0$  and  $|H_1(z)| < 0$  for arbitrary  $z \in \overline{U_0} \setminus \{p\}$ . We extend the local peak function  $H_1$  to a global peak function defined on  $M$  as follows. Choose a neighborhood  $V_1 \subset V_2$  of  $p$  and a constant  $c > 0$  such that  $\sup\{H_1(z) : z \in U_0 \cap V_2\} = -c$  and  $V_1 = \{z \in U_0 \cap V_2 : H_1(z) > -\frac{c}{2}\}$ .

$$H(z) = \begin{cases} H_1(z) & \text{if } z \in M \cap \overline{V_1} \\ \max(H_1(z), -\frac{3c}{2}) & \text{if } z \in M \cap (V_2 \setminus V_1) \\ -\frac{3c}{2} & \text{otherwise.} \end{cases}$$

Hence we can define the topology with the other method. Namely, by  $\{z \in M : H(z) > -\beta\}$  for arbitrary  $\beta > 0$ .

*Proof.* We will use the two topologies above to locate the images of holomorphic maps  $\phi_j$ . Now, let us show that if  $\phi_j(q) \rightarrow p$  as  $j \rightarrow \infty$ , then for arbitrary  $z \in M$ ,  $\phi_j(z) \rightarrow p$  as  $j \rightarrow \infty$ . Indeed, if  $\phi_j(z_0) \not\rightarrow p$ , we let  $\gamma(t)$  be a curved connected  $q$  and  $z_0$  such that  $\gamma(0) = q$  and  $\gamma(1) = z_0$ . We use a family of polydiscs  $\{\mathbb{D}_s^m\}_{s=1}^\tau$  with fixed radius  $r = (r_1, \dots, r_m)$  to cover  $\gamma$  such that the center  $o_s$  of  $\mathbb{D}_s^m$  is contained in  $\mathbb{D}_{s-1}^m(o_{s-1}, \frac{r}{2})$  and  $o_1 = q$ ,  $z_0 \in \mathbb{D}_\tau^m$ . As in [8] and [19], pick up  $j$  such that  $H(\phi_j(q)) > -a$  for some positive  $a$ . We consider

$$-a < H \circ \phi_j(q) \leq \left(\frac{1}{2\pi}\right)^m \int \cdots \int_{[0, 2\pi]^m} H(\phi_j(q + (r_1 e^{i\theta_1}, \dots, r_m e^{i\theta_m}))) d\theta_1 \cdots d\theta_m.$$

where the second inequality is because the sub-mean value inequality ( $H \circ \phi_j$  is plurisubharmonic so it is subharmonic). Let  $E_a = \{\theta \in [0, 2\pi]^m : H \circ \phi_j(q + r e^{i\theta}) > -2a\}$  and a  $\beta > 0$  depending on  $a$  such that  $\{z \in M : H(z) > -2^m a\} \subset \{z \in M :$

$S(z) < -\beta\}$  and  $\beta \rightarrow \infty$  as  $a \rightarrow 0$ . Hence, we have

$$-\alpha < \left(\frac{1}{2\pi}\right)^m \int_{E_a} \cdots \int 0 \, d\theta_1 \cdots d\theta_m + \left(\frac{1}{2\pi}\right)^m \int_{[0, 2\pi]^m \setminus E_a} \cdots \int -2^m a \, d\theta_1 \cdots d\theta_m$$

which implies  $\mu(E_a) < \pi^m$  where  $\mu$  is a Borel measure. However, when we apply the estimate  $\mu(E_a)$  to the Poisson integral of  $S \circ \phi_j$ , we have

$$\begin{aligned} & S \circ \phi_j(z) \\ & \leq \mathcal{K} \left(\frac{1}{2\pi}\right)^m \int_{[0, 2\pi]^m} \cdots \int S \circ \phi_j(q + (r_1 e^{i\theta_1}, \dots, r_m e^{i\theta_m})) \, d\theta_1 \cdots d\theta_m \\ & \leq \mathcal{K} \left(\frac{1}{2\pi}\right)^m \int_{[0, 2\pi]^m \setminus E_a} \cdots \int 0 \, d\theta_1 \cdots d\theta_m + \mathcal{K} \left(\frac{1}{2\pi}\right)^m \int_{E_a} \cdots \int -\beta \, d\theta_1 \cdots d\theta_m \\ & \leq -\left(\frac{1}{2}\right)^m \mathcal{K} \beta \end{aligned}$$

for all of  $z \in \mathbb{D}_1^m(o_1, \frac{r}{2})$ , where  $\mathcal{K}$  is the upper bound of the Poisson kernel for polydisc  $\mathbb{D}_1^m(o_1, \frac{r}{2})$ . One can now observe that if  $\phi_j(q) \rightarrow p$ , then  $\phi_j(z) \rightarrow p$  also for arbitrary  $z \in \mathbb{D}_1^m(o_1, \frac{r}{2})$ , because when  $a \rightarrow 0$ ,  $\beta \rightarrow \infty$ . Since  $o_2 \in \mathbb{D}_1^m(o_1, \frac{r}{2})$ ,  $\phi_j(o_2) \rightarrow p$  and we can prove by induction that  $\phi_j(z_0) \rightarrow p$ .

We will now prove that on each compact subset  $K \Subset M$ ,  $\{\phi_j\}_{j=1}^\infty$  has subsequence that converges uniformly. Suppose this is not true for a compact  $K$ . Then there exist  $z_j \in K$  such that  $\phi_j(z_j) \not\rightarrow p$  as  $j \rightarrow \infty$ . Since  $K$  is compact,  $z_j$  has limit point  $q'$ . Now we consider the polydisc  $\mathbb{D}^m(q', r)$  which covers most of  $z_j$ . By the discussion above, we know  $\phi_j(q') \rightarrow p$  as  $j \rightarrow \infty$ . For any small  $a > 0$  choose  $N > 0$  so that  $H \circ \phi_j(q') > -a$  for all of  $j > N$ . By the argument above, one can obtain  $S \circ \phi_j(z_j) < -\mathcal{K}(\frac{1}{2})^m \beta$ , where the  $\beta$  is determined by  $\alpha$ . It turns out that as  $j \rightarrow \infty$ , then  $S \circ \phi_j(z_j) \rightarrow -\infty$ . In other words,  $\phi_j(z_j) \rightarrow p$  which contradicts with our assumption.  $\square$

**Lemma 1.3.** *Let  $M$  be a simply-connected complete Kähler manifold with a negative sectional curvature. Suppose, also,  $M$  has a constant negative holomorphic sectional curvature in a neighborhood  $U$  of a point  $p \in \partial M$  at infinity in  $\overline{M}$ . Assume there is a family of biholomorphic maps  $\phi_j$  uniformly converges to  $p$  on compact subsets of  $M$ . Then  $M$  is a monotone union of subsets of  $\mathbb{B}^m$ , i.e.  $M = \bigcup_{j=1}^\infty B_j$ , where  $B_j$  is biholomorphic to some subset  $\mathcal{C}_j$  of  $\mathbb{B}^m$  for each  $j$  with  $\partial \mathcal{C}_j \cap \mathbb{S}^m \neq \emptyset$ .*

*Proof.* By Lemma 1.2,  $\phi_j$  uniformly converges to  $p$  on compact subsets of  $M$ . Given exhaustive compact subsets  $K_j \subset M$  such that  $K_j \subset K_{j+1}$  for each  $j$ , there exists  $\phi_{i_j}$  such that  $\phi_{i_j}(K_j) \subset U_0$  and  $\phi_{i_{j-1}}(K_j) \supset U_0$ . That is  $M \supset \phi_{i_j}^{-1}(U_0) \supset K_j \supset \phi_{i_{j-1}}^{-1}(U_0)$ . Let  $B_j$  be  $\phi_{i_j}^{-1}(U_0)$  and we can see  $M = \bigcup_{j=1}^\infty B_j$ . Moreover  $f \circ \phi_{i_j}(B_j) = f(U_0) = \mathcal{C}_j$  where  $\partial f(U_0) \cap \partial \mathbb{S}^m \neq \emptyset$ .  $\square$

**Lemma 1.4.** *Assume,  $M$  is a monotone union of subsets of  $\mathbb{B}^m$  as in Lemma 1.3. Then  $M$  admits a holomorphic imbedding into the unit balls  $\mathbb{B}^m$ .*

*Proof.* Write  $M = \bigcup_{j=1}^\infty B_j$ , with  $B_j$  biholomorphically equivalent to a subset  $\mathcal{C}_j$  of the unit ball via  $f_j$  and  $B_j \subset B_{j+1}$ , where  $\overline{\mathcal{C}_j}$  contains a piece of sphere. Fix  $z_0 \in B_1$ ,

and without loss of the generality, we assume  $f_j : B_j \rightarrow \mathbb{B}^m$  be a holomorphic embedding map with the non vanishing limit of  $\det(Jf_j)$  (the determinant of the Jacobian of  $f_j$ ). This is possible because we can always compose  $f_j$  with a Möbius transform  $h_j$  so that the limit of  $\det(J(h_j \circ f_j))(z_0)$  is nonzero and by Hurwitz theorem, it is nonzero everywhere (However the image of  $f_j$  might be various open subsets of  $\mathbb{B}^m$  now). Since the unit ball  $\mathbb{B}^m$  is taut, passing to a subsequence if necessary, then  $\{f_j\}_{j=1}^\infty$  converges uniformly on compact sets of  $M$  to a holomorphic map  $\phi : M \rightarrow \mathbb{B}^m$ . Since the limit of  $\det(Jf_j)$  is nonzero, we obtain a domain  $\phi(M) \subset \mathbb{B}^m$  which is  $m$ -dimensional.

It remains to prove the injectivity of  $\phi$ . Let  $z, w \in M$  such that  $\phi(z) = \phi(w)$ , then for arbitrary  $\epsilon > 0$ , there is  $N > 0$  so that for any  $j > N > 0$ ,  $d_{\mathbb{B}^m}(f_j(z), f_j(w)) < \epsilon$  ( $\mathbb{B}^m$  and later on, the  $\mathcal{B}^m$  are considered as balls with a complete constant negative curved Kähler metrics). And we choose one small ball  $\mathcal{B}^m \subset \phi(M)$  independent of  $j$  contains  $f_j(z)$  and  $f_j(w)$ . With a generalized Schwarz lemma of Yau [22] for  $f_j^{-1}$ , we have

$$(1) \quad d_M(z, w) \leq \frac{K_1}{K_2} d_{\mathcal{B}^m}(f_j(z), f_j(w)) < \frac{K_1}{K_2} \epsilon',$$

where  $K_1$  is the lower bound of Ricci curvature of  $\mathcal{B}^m$ ,  $K_2$  is the upper bound of holomorphic bisectional curvature of a compact submanifold including  $z$  and  $w$  and  $\epsilon'$  is the upper bound of  $d_{\mathcal{B}^m}(f_j(z), f_j(w))$ .  $K_2 < 0$  exists because, for each pair  $z, w$ , there exists a compact submanifold of  $M$  such that the sectional curvature are bounded above by a negative constant. Thus  $\phi(z) = \phi(w)$  will imply  $d_M(z, w) = 0$  which gives the proof of the injectivity.  $\square$

## 2. COROLLARIES

We now prove the following corollary which can be seen as a Wong-Rosay type theorem in Kähler manifolds.

**Corollary 2.1.** *Let  $M$  be a simply-connected complete Kähler manifold with a negative sectional curvature. Suppose, also,  $M$  has a constant negative holomorphic sectional curvature in a neighborhood  $U$  of a point  $p \in \partial M$  at infinity in  $\overline{M}$ . If there is a family of biholomorphic maps  $\phi_j$  so that for any  $q \in M$ ,  $\phi_j(q) \rightarrow p$ , as  $j \rightarrow \infty$ , then  $M$  is biholomorphic to the unit ball  $\mathbb{B}^m$*

*Proof.* Since  $M$  is holomorphic embedding into a unit ball  $\mathbb{B}^m$ , and by [18] the chart of the asymptotic boundary around  $p$  is not only homeomorphism with but also CR diffeomorphism with a part of  $\partial \mathbb{B}^m$  which is strongly pseudoconvex. (This implies also the asymptotic boundary coincide with regular boundary of a domain.) Thus,  $p$  is a strongly pseudoconvex point. Then the corollary follows by the original Wong-Rosay theorem.  $\square$

*Remark 2.* Alternatively, the last corollary can be proved by the following argument. Since  $M$  is taut by Theorem 0.1, we can see  $\{f_j^{-1}\}$  is also a normal family. We define  $F$  to be the limit of  $f_j$  and  $G$  the limit of  $f_j^{-1}$ . Due to  $\text{Id} = f_j \circ f_j^{-1} = f_j^{-1} \circ f_j$ , we obtain  $F \circ G = G \circ F = \text{Id}$ . Thus  $M$  is biholomorphic to a subset of  $\mathbb{B}^m$  such that  $F(\partial M) \subset \partial \mathbb{B}^m$ . By an argument of [17], the proof is complete.

Also the following corollary is immediately from Theorem 0.1.

**Corollary 2.2.**  *$M$  is taut.*

**Acknowledgments.** I thank a lot my advisor Prof. Steven Krantz for always being patience to answer my any (even stupid) question. He also encourages me a lot for the career and the life as both of a supervisor and a raconteur. I also thank Prof. Quo-Shin Chi who taught me various geometries and spent much time with me (to help me understand Ricci flow, algebraic geometry and some other interesting topics). Last but not least, I appreciate Prof. Edward Wilson for his kindness and consistent support since I arrived in USA. I always have much profited from the discussion between us.

## REFERENCES

- [1] Werner Ballmann, Mikhael Gromov, and Viktor Schroeder, *Manifolds of nonpositive curvature*, Progress in Mathematics, vol. 61, Birkhäuser Boston Inc., Boston, MA, 1985. MR823981 (87h:53050)
- [2] Steve Bell and Ewa Ligocka, *A simplification and extension of Fefferman's theorem on bi-holomorphic mappings*, Invent. Math. **57** (1980), no. 3, 283–289. MR568937 (81i:32017)
- [3] François Berteloot, *Sur certains domaines faiblement pseudoconvexes dont le groupe d'automorphismes analytiques est non compact*, Bull. Sci. Math. **114** (1990), no. 4, 411–420. MR1077269 (91j:32035)
- [4] Jeff Cheeger and David G. Ebin, *Comparison theorems in Riemannian geometry*, AMS Chelsea Publishing, Providence, RI, 2008. Revised reprint of the 1975 original. MR2394158 (2009c:53043)
- [5] Wing Sum Cheung, Siqi Fu, Steven G. Krantz, and Bun Wong, *A smoothly bounded domain in a complex surface with a compact quotient*, Math. Scand. **91** (2002), no. 1, 82–90. MR1917683 (2003d:32028)
- [6] P. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific J. Math. **46** (1973), 45–109. MR0336648 (49 #1421)
- [7] Hervé Gaussier, Kang-Tae Kim, and Steven G. Krantz, *A note on the Wong-Rosay theorem in complex manifolds*, Complex Var. Theory Appl. **47** (2002), no. 9, 761–768. MR1925173 (2003f:32025)
- [8] Robert E. Greene, Kang-Tae Kim, and Steven G. Krantz, *The geometry of complex domains*, Progress in Mathematics, vol. 291, Birkhäuser Boston, Inc., Boston, MA, 2011. MR2799296 (2012c:32001)
- [9] Robert E. Greene, Kang-Tae Kim, Steven G. Krantz, and Aeryeong Seo, *Semicontinuity of automorphism groups of strongly pseudoconvex domains: the low differentiability case*, Pacific J. Math. **262** (2013), no. 2, 365–395. MR3069066
- [10] A. V. Isaev and S. G. Krantz, *Domains with non-compact automorphism group: a survey*, Adv. Math. **146** (1999), no. 1, 1–38. MR1706680 (2000i:32053)
- [11] Alexander V. Isaev and Steven G. Krantz, *Characterization of Reinhardt domains by their automorphism groups*, J. Korean Math. Soc. **37** (2000), no. 2, 297–308. Several complex variables (Seoul, 1998). MR1775961 (2001i:32032)
- [12] Alexander V. Isaev and Steven G. Krantz, *On the automorphism groups of hyperbolic manifolds*, J. Reine Angew. Math. **534** (2001), 187–194. MR1831637 (2002b:32033)
- [13] Wilhelm Kaup, *Some remarks on the automorphism groups of complex spaces*, Rice Univ. Studies **56** (1970), no. 2, 181–186 (1971). MR0274805 (43 #565)
- [14] Kang-Tae Kim and Steven G. Krantz, *A Kobayashi metric version of Bun Wong's theorem*, Complex Var. Elliptic Equ. **54** (2009), no. 3-4, 355–369. MR2513545 (2010g:32040)
- [15] Paul F. Klembeck, *Kähler metrics of negative curvature, the Bergmann metric near the boundary, and the Kobayashi metric on smooth bounded strictly pseudoconvex sets*, Indiana Univ. Math. J. **27** (1978), no. 2, 275–282. MR0463506 (57 #3455)
- [16] Akio Kodama, Steven G. Krantz, and Daowei Ma, *A characterization of generalized complex ellipsoids in  $\mathbb{C}^n$  and related results*, Indiana Univ. Math. J. **41** (1992), no. 1, 173–195. MR1160908 (93d:32033)
- [17] Walter Rudin, *Holomorphic maps that extend to automorphisms of a ball*, Proc. Amer. Math. Soc. **81** (1981), no. 3, 429–432. MR597656 (82c:32012)

- [18] Harish Seshadri and Kaushal Verma, *A class of nonpositively curved Kähler manifolds bi-holomorphic to the unit ball in  $\mathbb{C}^n$* , C. R. Math. Acad. Sci. Paris **342** (2006), no. 6, 427–430. MR2209223 (2006m:32023)
- [19] Nessim Sibony, *A class of hyperbolic manifolds*, Recent developments in several complex variables (Proc. Conf., Princeton Univ., Princeton, N. J., 1979), 1981, pp. 357–372. MR627768 (83a:32022)
- [20] Yum Tong Siu and Shing Tung Yau, *Complete Kähler manifolds with nonpositive curvature of faster than quadratic decay*, Ann. of Math. (2) **105** (1977), no. 2, 225–264. MR0437797 (55 #10719)
- [21] Kaushal Verma, *A characterization of domains in  $\mathbb{C}^2$  with noncompact automorphism group*, Math. Ann. **344** (2009), no. 3, 645–701. MR2501305 (2009m:32040)
- [22] Shing Tung Yau, *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math. **100** (1978), no. 1, 197–203. MR0486659 (58 #6370)

DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY, SAINT LOUIS, USA  
E-mail address: `bingyuan@math.wustl.edu`